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Resonant trapping: A failure mechanism in switch transitions

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Stochastic resonance in a time-modulated bistable system is shown to survive at modulation amplitudes larger than the relevant dynamical bistability threshold. Residual stochastic resonance is related to a synchronization-loss mechanism (*resonant trapping*), which attains its maximum effect for noise-intensities proportional to the excess modulation amplitude. Such a phenomenon provides an effective model of noise-induced failures in switch devices. [S1063-651X(97)03401-6]

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The notion of stochastic resonance (SR) was originally introduced [1] to describe a property peculiar to bistable systems subjected to a *weak* periodic modulation $A(t)=A_0\cos(\Omega t)$ embedded into a noise background $\xi(t)$: for a certain value of the noise intensity D the fundamental periodic component of the system response gets amplified to an optimal amplitude proportional to A_0 [2]. Bistability is generally deemed to be a key ingredient of SR [2–4]. In the present paper we explore the effects of *strong* modulating signals $A_0>A_{\text{th}}$, where A_{th} denotes the system bistability threshold, which is the critical value of A_0 where one of the two stable states disappears.

Bistable systems used as switches are a fundamental component of electronic and optical devices [5,6]. A detailed understanding of the role of spurious noises in the switching dynamics of a bistable system is thus of great technological interest. Hysteretic devices (electronic triggers) have been long employed to simulate standard SR [2]. However, in the operating conditions investigated here, the switch device is driven by an above-threshold sinusoidal signal with $A_0>A_{\text{th}}$ and noise is fed in to mimic unspecified weak disturbances, which are likely a cause of failure in the switch performance.

We anticipate the main conclusions of our study. (a) Resonant amplification of the periodic components of the system response at optimal D values may be observed for $A_0>A_{\text{th}}$ too (*residual* SR). (b) The switch time distributions show a peak structure with maxima located at the odd multiples of half the modulation period $T_\Omega=2\pi/\Omega$. All peaks but the first one attain their maximum for *one* value of the noise intensity. This property may be attributed to a different synchronization-loss (or failure) mechanism induced by noise (*resonant trapping*). (c) The trapping phenomenon is strongly enhanced by color, whereas standard SR is suppressed by increasing the noise correlation time. We conclude that a noisy bistable system operating in the hysteretic regime is characterized by entirely different synchronization properties.

As in Refs. [1] and [2], we simulated by means of an analog circuit the quartic bistable process

$$\dot{x}=-V'(x)+\xi(t)+A_0\cos(\Omega t), \tag{1}$$

with $V(x)=-ax^2/2+bx^4/4$. Here $\pm x_m=\sqrt{a/b}$ denote the potential minima, $\Delta V=a^2/4b$ is the potential barrier, and $A_{\text{th}}=2ax_m/3\sqrt{3}$ is the static bistability threshold. Our noise generator produces a random signal $\xi(t)$ with zero mean value, Gaussian distribution, and autocorrelation function $\langle\xi(t)\xi(0)\rangle=(D/\tau_c)\exp(-|t|/\tau_c)$. The correlation time τ_c can be set so small ($a\tau_c=0.02$) that $\langle\xi(t)\xi(0)\rangle$ approximates a δ function (white noise limit).

For vanishingly weak noise intensities $D\simeq 0$ and modulation amplitudes A_0 larger than A_{th} [5] the process (1) develops dynamical hysteresis. Namely, when the external signal $A(t)$ is periodically modulated, instead of being adiabatically changed ($\Omega=0$), the system exhibits a delay in switching state; i.e., a switch event occurs for a value of $A(t)$ larger than the static threshold A_{th}. Consequently, a dynamical threshold $A_c(\Omega)$ may be defined as the critical A_0 value above which deterministic switch events take place, driven by the periodic signal alone. The dynamical threshold $A_c(\Omega)$ depends on both the modulation frequency and the wave form of $A(t)$. For the sinusoidal input signal of Eq. (1) we predict that

$$A_c(\Omega)/A_{\text{th}}=1+\beta(\Omega/a), \tag{2}$$

where $\beta\simeq(2\sqrt{3}/\pi)g_1$ and the smallest zero g_1 of the Airy function $\text{Ai}'(-x)$ is order unity [7]. Prediction (2) follows immediately the analytical treatment of Ref. [5]. Our analytical estimate $\beta\simeq 1.10$ compares very closely with the simulation fitting constant $\beta\simeq 1.05\pm 0.10$ [8]. Throughout our

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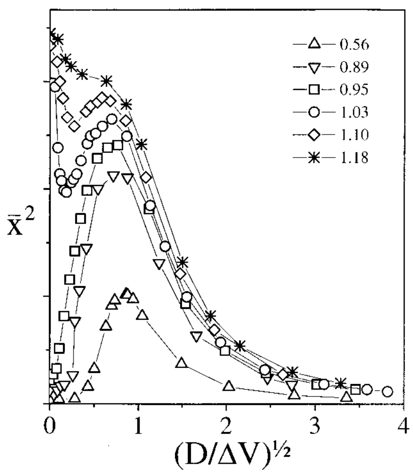

FIG. 1. $\overline{x(D)}$ (arbitrary units) vs D for the process (1) with $a = 4.8 \times 10^3\ \mathrm{s}^{-1}$, $\nu_\Omega = 102$ Hz, and different values of $z(A_0)$. Here $z(A_0) = A_0/A_c(\Omega) - 1$ and $A_c(\Omega) = 1.15 A_{\mathrm{th}}$.

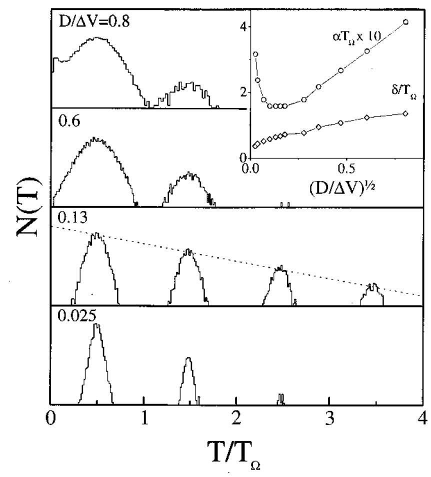

FIG. 2. Switch time distributions $N(T)$ for $A_0 = 1.03 A_c(\Omega)$ and different values of D. Other parameter values are the same as in Fig. 1. Inset: the fitting parameters α and δ as a function of D. The envelope curve $\exp(-\alpha T)$ is dotted for the reader's convenience.

simulation work the modulation frequency $\nu_\Omega = \Omega/2\pi$ was kept constant, well within the range of validity of Eq. (2).

We started our investigation by plotting in Fig. 1 the amplitude $\overline{x(D)}$ of the fundamental periodic component of the process $x(t)$ as a function of the noise intensity at increasing A_0 values. For $A_0 < A_c(\Omega)$ the curve $\overline{x(D)}$ exhibits the well-known SR profile with one absolute maximum. On raising A_0 larger than the dynamical threshold $A_c(\Omega)$ and keeping D very small, the switch events are mostly driven by the external driving signal as testified by the overshoot of $\overline{x(D)}$ at $D = 0$. However, a residual SR mechanism is apparently still at work: a SR peak (relative maximum) continues to exist for A_0 larger than but close to $A_c(\Omega)$ and eventually merges into the decaying background branch of driven switches for even larger A_0 values. On looking at the position and magnitude of the SR peaks one is led to conclude that SR and driven switching dynamics coexist in the regime of dynamical hysteresis, at variance with common wisdom that bistability is a necessary SR ingredient. One might interpret residual SR as due to the fact that over one forcing cycle the system alternates time intervals when dynamical bistability is preserved, i.e., $A(t) < A_c(\Omega)$, and shorter temporal windows when, for $A(t) > A_c(\Omega)$, switches are driven deterministically by the forcing signal. In the former case random interwell switches can be activated by the noise $\xi(t)$ alone, on an intrinsic time scale $T_K(D)$ [whence comes the synchronization condition of standard SR [9] $T_K(D) = \pi/2\Omega$]. Such a naive explanation was disproved by replacing the sinusoidal signal $A(t)$ of Eq. (1) with a square wave with equal amplitude and period. The relevant $\overline{x(D)}$ curves generated by our analog simulator are indistinguishable from those reported in Fig. 1, the only difference being a lowering of the dynamical threshold $A_c(\Omega)$ [with $\beta = 0.3$ in Eq. (2)].

Residual SR has nothing to do with the bistable dynamics of the undriven stochastic system (1), but is rather the signature of a failure mechanism in the driven switching dynamics. Chances are that $x(t)$ is retarded by noise with respect to its deterministic trajectory and that the relevant delay time is of the order of $T_\Omega/2$: $x(t)$ thus gets trapped in the unstable potential well and the switch event is postponed, with exponentially decreasing probability, by an integer multiple of the forcing period. As a consequence, a failure occurs in the switching dynamics, resulting in the local minimum in the $\overline{x(D)}$ curves. On further increasing D the trapping mechanism becomes ineffective: the noise itself allows $x(t)$ to be swept into the instantaneously stable potential well. Such a desynchronization mechanism is termed here *resonant trapping* (RT). Trapping in a metastable potential well was advocated in Refs. [10–12] to explain why the relevant nonstationary escape times attain a maximum for an optimal value of D.

A deeper insight into RT was gained by monitoring the switch time distributions $N(T)$. Here T denotes the time interval between two subsequent switches (or barrier crossings) in opposite directions [3,9]. All $N(T)$ curves in Fig. 2 are characterized by a peak structure with maxima at $T_n = (n - 1/2) T_\Omega$. The peak heights $N(T_n)$ decrease to a good approximation according to the exponential law $N(T_{n+1})/N(T_n) = \exp(-\alpha T_\Omega)$ with α a function of D. Moreover, for A_0 close to the dynamical threshold $A_c(\Omega)$ the profile of each individual $N(T)$ peak fits well a Gaussian function with standard deviation δ proportional to $T_\Omega D^\gamma$, $\gamma \simeq 0.15 \pm 0.05$. Thus the *normalized* switch time distribution $N(T)$ is determined by the two phenomenological parameters α and δ, both D dependent [3]. The differences between residual and standard SR are remarkable indeed. (i) At very low noises, only the first $N(T)$ peak is detectable for $A_0 > A_c(\Omega)$, as expected, since the system is driven by a (weakly noisy) above-threshold signal. Note that for $A_0 < A_c(\Omega)$ and low noise, a large number of peaks would be visible with envelope constant α proportional to $\sqrt{D}$ [3]. (ii) On increasing D we reach a resonance condition when all peaks of $N(T)$ with $n \geqslant 2$ (Fig. 2) attain their maximum height simultaneously or, equivalently, $\alpha(D)$ approaches a lower bound. Due to the normalization constraint, such a condition corresponds to a minimum of the first peak. We remember that standard SR requires that the first $N(T)$ peak dominates over both the remaining peaks and the random switch background [9]. (iii) On further increasing D the

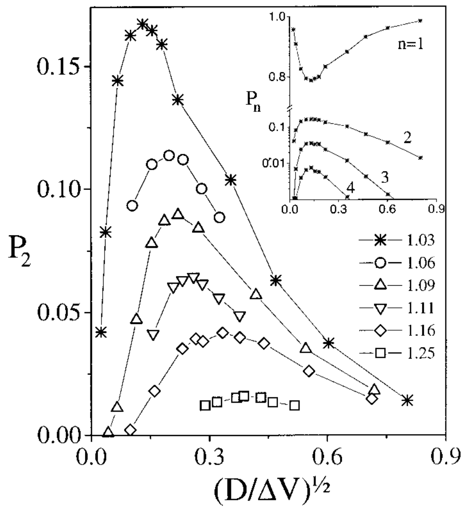

FIG. 3. $P_2(D)$ for different values of $z(A_0)$. Inset: $P_n(D)$ with $1 \leqslant n \leqslant 4$ for $A_0 = 1.03 A_c(\Omega)$. Other parameter values are the same as in Fig. 1.

peaks with $n \geqslant 2$ disappear again. Correspondingly, and at odds with standard SR, the first peak in Fig. 1 recovers, broadens, and overlaps with an exponential background that eventually washes out the entire peak structure.

Properties (i)–(iii) are summarized in Fig. 3, where the peak strength $P_n(D)$ (i.e., the integral of $N(T)$ over the interval $[T_n - T_\Omega/4,\ T_n + T_\Omega/4]$) is plotted as a function of D for different values of A_0: the resonance condition for all $P_n(D)$ with $n \geqslant 2$ is established at $D = D_{\rm RT}$, simultaneously, in coincidence with a minimum of $P_1(D)$. Most importantly, the maxima of $P_2(D)$ shift towards higher $D_{\rm RT}$ values with an increase in the modulation amplitude and a decrease in magnitude. On comparing Figs. 1 and 3 we notice that $D_{\rm RT}$ closely locates the dips that separate the maxima at $D=0$ from the residual SR peaks in the $\overline{x(D)}$ curves. The quantities $D_{\rm RT}$ and $P_2(D_{\rm RT})$ are plotted versus $A_0 - A_c(\Omega)$ in Fig. 4 for the purpose of a quantitative analysis.

Far from attempting to develop a full analytical theory for the process under study, we limit ourselves to a qualitative interpretation of the simulation results of Figs. 3 and 4. First of all, we derive a heuristic law for the dependence of $D_{\rm RT}$ on A_0. Since we are working just above the dynamical threshold $A_c(\Omega)$ we can follow the approach of Ref. [5]. The effective potential $V(x) - xA(t)$ develops a horizontal flexural point at $x_{\rm th} = -x_m/\sqrt{3}$ and the time evolution of $x(t)$ around this point is approximated by

$$\dot{y} = (a/x_{\rm th}) y^2 + \overline{A} \sin(\Omega t) + \xi(t), \tag{3}$$

where $y(t) = x(t) - x_{\rm th}$, $\overline{A^2} = A_0^2 - A_c^2(\Omega)$, and Ω is assumed to be much smaller than the intrinsic rate $|V''(0)| = a$. An estimate of the trapping time τ_T, i.e., the delay time of $y(t)$, can be obtained by equating the *free diffusion* induced by noise in the vicinity of the horizontal flexural point $x_{\rm th}$ and the square *ballistic* displacement driven by the effective forcing term (3) with amplitude $\overline{A}$, that is, $2D\tau_T = (\overline{A}/\Omega)^2 \sin^2(\Omega\tau_T)$. At resonance the total delay (in both temporal directions) $2\tau_T$ comes close to the half modulation period; on imposing $\tau_T = \pi/\Omega$ for $A_0 \gtrsim A_c(\Omega)$, simple algebraic manipulations yield our estimate of $D_{\rm RT}$,

$$\frac{D_{\rm RT}}{\Delta V} = \frac{1}{\pi}\left(\frac{2}{3}\right)^3 \frac{a}{\Omega} z(A_0), \tag{4}$$

with $z(A_0) \equiv A_0/A_c(\Omega) - 1$. The slope of the linear fit in Fig. 4 is thus reproduced within a 10% accuracy.

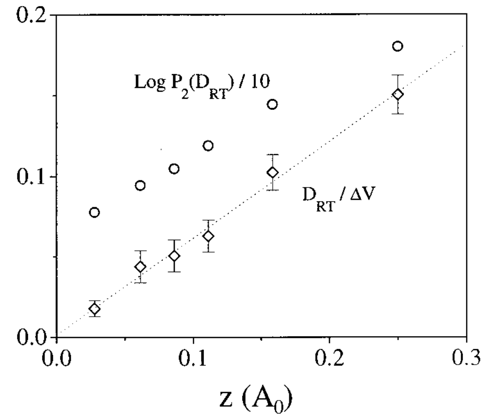

FIG. 4. $\log_{10}\ P_2(D_{\rm RT})$ (circles) and $D_{\rm RT}$ (lozenges) vs $z(A_0)$. The straight line (with slope 0.6) is the least-squares fit of Eq. (4) for $D_{\rm RT}$.

We know from Fig. 2 that for $A_0 \gtrsim A_c(\Omega)$ the switch time distribution $N(T)$ can be modeled as a sum of Gaussian functions centered at T_n, that is, $N(T) = (1-\mu)\sum_{n=1}^{\infty} \mu^{n-1} G(T - T_n, \delta)$, where $\mu \equiv \exp(-\alpha T_\Omega)$ and $G(x,\delta)$ denotes a normalized Gaussian distribution with zero mean and variance δ^2. The experimental evidence that all $N(T)$ peaks with $n \geqslant 2$ hit their maximum for the same value of $D_{\rm RT}$ of the noise intensity implies that with increasing D the envelope constant α decreases down to an optimal value $\alpha(D_{\rm RT})$, such that $\alpha(D_{\rm RT}) T_\Omega > -\ \ln 2$, and then increases again for $D > D_{\rm RT}$. Correspondingly, the strength of the second peak reads $P_2 = \mu(1-\mu)$ and $P_2(D_{\rm RT}) < 0.25$ [13].

Finally, we observed that RT is quite sensitive to the noise correlation time τ_c. This conclusion is well supported by the $P_2(D)$ curves for different color regimes (and fixed modulation amplitude) [13]. The resonant behavior of $P_2(D)$ is *enhanced* by increasing τ_c and the RT condition is achieved at higher noise intensities. The shift of the P_2 peaks towards larger D values follows immediately the perturbation result [14] that color-induced dynamical effects in process (1) are reproduced by simply rescaling D into $D/(1 + a\tau_c)$; hence estimate (4) for $D_{\rm RT}$ must be corrected by the multiplicative factor $1 + a\tau_c$, in quantitative agreement with the outcome of our analog simulation [13]. For the sake of comparison, we mention here that, at odds with the present case, standard SR is depressed by color [15]. In conclusion, noise-induced failures of a switch device driven by an above-threshold signal are interpreted in terms of a resonant mechanism, named here resonant trapping, whose properties, in many ways, are opposite to those of standard SR.

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